3.4: The Derivative as a Rate of Change

Def^n: The instantaneous rate of change of f at x_0 is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Ex: (1) The area of a circle is $A = \pi r^2 = \cancel{\frac{\pi D^2}{4}}$ where r = radius ~~and D=diameter~~. How fast does area change with respect to the radius where radius is 5m?

$$\frac{dA}{dr} = 2\pi r \quad \text{so} \quad \left.\frac{dA}{dr}\right|_{r=5} = 10\pi \ m^2/m \approx 31.42 \ m^2/m$$

(meters squared per meter)

~~Thm~~ Let $s := s(t)$ be a function measuring distance traveled for some object. Then the displacement of the object in the interval t to $t+\Delta t$ is given by $s(t+\Delta t) - s(t)$. The average velocity over that time interval is

$$v_{av} := \frac{\text{displacement}}{\text{travel time}} = \frac{s(t+\Delta t) - s(t)}{\Delta t}.$$

As $\Delta t \to 0$, we get

Def^n: The velocity (instantaneous velocity) is the derivative of position with respect to time.

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t} = \frac{\text{units of distance}}{\text{units of time}}.$$

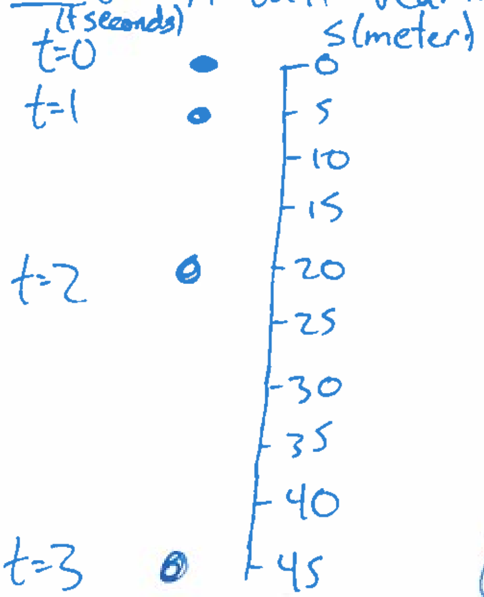
The speed of the object is

$$\text{speed} = |v(t)| = \left|\frac{ds}{dt}\right|.$$

The acceleration (instantaneous acceleration) is the derivative of velocity with respect to time

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{\text{units of distance}}{(\text{units of time})^2}$$

Ex:(2) A ball bearing begins free fall at time $t=0$.



(a) How many meters does the bearing fall in 3 seconds?

(b) Given that $s(t)=4.9t^2$, what is the speed, velocity and acceleration at $t=3$ seconds

(a) 45 meters

(b) $v(t)=9.8t$, $a(t)=9.8$

$v(3)=29.4\ m/sec$, speed $=|v(3)|=29.4\ m/sec$

$a(3)=9.8\ m/sec^2$.

Ex(3): A dynamite blast blows a rock straight up with a velocity of 160 ft/sec. It reaches height $h(t)=160t-16t^2$ ft after t seconds.

(a) How high does the rock go?

(b) What are velocity and speed at 256 ft above ground? Going Up? Going Down?

(c) What is the acceleration at time t seconds?

(d) When does the rock hit the ground?

(a) $v(t)=0=160-32t \Rightarrow t=5$ seconds. and $h(t)=400$ ft

(b) $h(t)=256$ then $t=2,8$ seconds $v(2)=96\ ft/sec$ $v(8)=-96\ ft/sec$.

(c) $a(t)=-32\ ft/sec^2$

(d) $h(t)=0$ then $t=0,10$ seconds. Clearly hits the ground after 10 seconds.

Derivatives in Economics

Cost, Revenue and Profit fcns. $P = R - C$.

Derivatives are called marginal cost, revenue and profit.

Ex(4): Suppose cost of a product is given by

$$c(x) = x^3 - 6x^2 + 15x. \quad (x = \text{quantity} / c = \$)$$

Revenue is $r(x) = x^3 - 3x^2 + 12x \quad (x = \text{quantity} / r = \$)$

You currently produce and sell 10 radiators a day. What is your Additional cost, revenue and profit for producing and selling 1 more radiator per day.

Solution: $c'(x) = 3x^2 - 12x + 15 \Rightarrow c'(10) = \$195/\text{unit}$, costs \$195 to produce 1 more radiator

$r'(x) = 3x^2 - 6x + 12 \Rightarrow r'(10) = \$252/\text{unit}$ make \$252 selling 1 more radiator.

$p'(x) = r'(x) - c'(x) = 6x - 3 \Rightarrow p'(10) = r'(10) - c'(10)$
$= \$57/\text{unit}$ make \$57 selling one more radiator

3.6: The Chain Rule

The chain rule allows us to take derivatives of composite functions.

Ex(1): Identifying Composite fcns.

For each of the following, write the function as a composite fcn.

(a) $y=(3x^2+1)^2 \Rightarrow y=u^2$ where $u=3x^2+1$

(b) $f(x)=\ln(x^2+1)$. Then $f(x)=f(u)=\ln(u)$ where $u=x^2+1$.

(c) $y=e^{x^3} \Rightarrow y=e^u$ where $u=x^3$.

Chain Rule: If $f(u)$ is differentiable at $u=g(x)$ and $g(x)$ is differentiable at x, then

$$(f\circ g)'(x)=f'(g(x))\cdot g'(x)$$

or if $y=f(u)$ and $u=g(x)$ then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Proof: Let $\Delta u=g(x+\Delta x)-g(x)$ and $\Delta y=f(u+\Delta u)-f(u)$.

Then $\displaystyle \frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$ (as $\Delta x\to 0$, $\Delta u\to 0$)

$$=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$$

$$=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Ex(2): Take the derivative of each composite fcn from Ex (1).

(a) $y=(3x^2+1)^2=u^2$ where $u=3x^2+1$.

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=2u\cdot(6x)=2(3x^2+1)\cdot 6x=36x^3+12x$$

Try this by expanding $(3x^2+1)^2$ first and You will get the same answer.

(b) $f(x)=\ln(x^2+1)=\ln(u)$ where $u=x^2+1$.

$$f'(x)=\frac{1}{u}\cdot(2x)=\frac{1}{x^2+1}\cdot 2x=\frac{2x}{x^2+1}$$

(c) $y=e^{x^3}=e^u$ where $u=x^3$.

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u\cdot 3x^2=e^{x^3}\cdot 3x^2.$$

Ex(3): An object moves along the x-axis so that its position $(t\geq 0)$ is given by $x(t)=\cos(t^2+1)$. Find the velocity as a fcn of t.

$$v(t)=\frac{dx}{dt}=\frac{dx}{du}\cdot\frac{du}{dt} \quad \text{where } u=t^2+1.$$

So $v(t)=-\sin(u)\cdot 2t=-2t\cdot\sin(t^2+1)$.

Ex(4): Differentiate $\underbrace{\sin(\underbrace{x^2+e^x}_{\text{inside}})}_{\text{outside}}$. inside $=g(x)=x^2+e^x$

outside $=f(x)=\sin(x)$.

$$\frac{d}{dx}\left(\sin(x^2+e^x)\right)=f'(g(x))\cdot g'(x)=\cos(g(x))\cdot(2x+e^x)=\cos(x^2+e^x)(2x+e^x).$$

Repeated Use:

Ex(5): Let $g(t)=\tan(5-\sin(2t))$. Outside $=\tan(t)$
inside $=5-\sin(2t)$.

$g'(t)=\sec^2(5-\sin(2t))\cdot\frac{d}{dt}(5-\sin(2t))$. outside $=\sin t$
inside $=2t$

$=\sec^2(5-\sin(2t))\cdot(-\cos(2t)\cdot 2)=-2\cos(2t)\cdot\sec^2(5-\sin(2t))$.

Power Chain Rule:

$\frac{d}{dx}(u^n)=n\cdot u^{n-1}\cdot\frac{d}{dx}(u)$.

Ex(6): (a) $\frac{d}{dx}\left((5x^3-x^4)^7\right)=7(5x^3-x^4)^6(15x^2-4x^3)$

(b) $\frac{d}{dx}(\sin^5 x)=5\sin^4 x\cdot\cos x$.

(c) $\frac{d}{dx}\left(\frac{1}{(1-2x)^3}\right)=\frac{d}{dx}\left((1-2x)^{-3}\right)=-3(1-2x)^{-4}\cdot 2=\frac{-6}{(1-2x)^4}$